Isoperimetry of waists and local versus global asymptotic convex geometries

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1 Introduction

Theorem 1.1. *Assume that two symmetric convex bodies K and L in $\mathbb{R}^n$ have sections of dimensions at least k and $n-ck$ respectively whose diameters are bounded by 1. Then for a random orthogonal operator $U \in \mathcal{O}(n)$ the body $K \cap UL$ has diameter bounded by $C^{n/k}$ with probability at least $1-e^{-n}$.*

Here and thereafter $C, C_1, c, c_1, \dots$ denote positive absolute constants.

1. The main point of Theorem 1.1 is that the existence implies randomness in it. Namely, we do not assume that the two sections are random; their *existence* suffices. Yet the conclusion holds for a *random* rotation U. This seems to be a new phenomenon in the asymptotic convex geometry. It is further manifested by Corollary 1.2.

2. Theorem 1.1 is a local to global implication in the asymptotic convex geometry, see [MS]. The *local* information about K and L (the existence of bounded sections) implies the *global* information (bounded intersections of the whole bodies). This is further illustrated in Corollary 1.3.

3. The exponential bound $C^{n/k}$ in Theorem 1.1 can be improved to a polynomial bound, say $C(n/k)^2$, at the cost of decreasing the probability from $1-e^{-n}$ to $1-e^{-k}$. This will be proved in the Appendix by Mark Rudelson and the author.

We will first discuss two applications of Theorem 1.1 and then turn to the method used in its proof, which is rather general and whose main ingredient is the recent "isoperimetry of waists" due to M.Gromov.

The first immediate consequence of Theorem 1.1 is a striking statement "existence implies randomness" about the diameters of sections of symmetric convex bodies K:

"If K has a nicely bounded section,
then most sections of K are nicely bounded"

(with a certain loss of the diameter as well as of the dimension). This phenomenon was discovered by A.Giannopoulos, V.Milman and A.Tsolomitis in their forthcoming work [GMT], and independently by the author a few weeks later. Precisely, with $L = \mathbb{R}^{ck}$ or $K = \mathbb{R}^{n-k}$ in Theorem 1.1 one immediately obtains

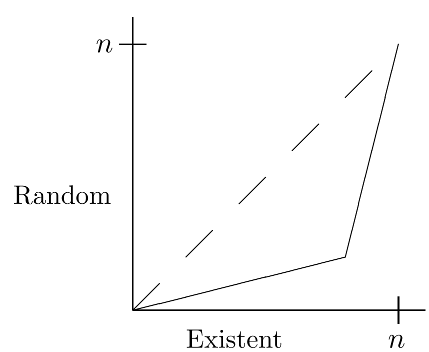
Corollary 1.2 (Propagation of boundedness of sections). *Let K be a convex symmetric body in $\mathbb{R}^n$ and k be a positive integer.*

(i) If there **exists** *a section of K of dimension k whose diameter is bounded by 1, then a* **random** *section of K of dimension ck has diameter bounded by $C^{n/k}$ with probability at least $1 - e^{-cn}$.*

(ii) If there **exists** *a section of K of dimension $n-ck$ whose diameter is bounded by 1, then a* **random** *section of K of dimension $n-k$ has diameter bounded by $C^{n/k}$ with probability at least $1 - e^{-cn}$.*

The randomness here is with respect to the Haar measure on the Grassmanian $G_{n,m}$.

The forthcoming paper [GMT] offers a more direct approach to this corollary as well as better bounds on the diameter (note also that the version of Theorem 1.1 in the Appendix gives polynomial bounds).



Dimensions of the existent and of the random sections in Corollary 1.2

The second application is that Theorem 1.1 can turn various local results in the asymptotic convex geometry into global statements. Let us show this on the example of the volume ratio theorem, one of the important "local" results in the field. A convex set K in $\mathbb{R}^n$ has the volume ratio A with respect to the unit Euclidean ball D if $D \subseteq K$ and $(|K|/|D|)^{1/n} = A$.

Corollary 1.3 (Global volume ratio theorem). *Assume that a convex set K in $\mathbb{R}^n$ has volume ratio A with respect to the unit Euclidean ball. Assume that a convex symmetric set L in $\mathbb{R}^n$ has a section of dimension k whose diameter is bounded by 1. Then for a random orthogonal operator $U \in \mathcal{O}(n)$ the body $K \cap UL$ has diameter bounded by $(2A)^{Cn/k}$ with probability at least $1 - e^{-n}$.*

For $L = \mathbb{R}^{n-k}$, Corollary 1.3 is the classical volume ratio theorem due to S.Szarek and N.Tomczak-Jaegermann (see e.g. [P]); the best constant in this case is known to be $C = 1$ (with 4π replacing the factor of 2).

Proof. By Rogers-Shephard [RS], the volume of $K' = K - K$ is $|K'| \le \binom{2n}{n}|K| \le 4^n|K|$. Then K' is symmetric and its volume ratio with respect to the Euclidean ball is at most $4A$. By the volume ratio theorem (see e.g. [P]), K' has a section of dimension at least $n - ck$ whose diameter is bounded by $M = (4\pi A)^{n/ck}$. The proof is finished by applying Theorem 1.1 to $M^{-1}K'$ and L. ■

Our approach to the proof of Theorem 1.1 is based on a recent isoperimetric theorem of M.Gromov [G], his "isoperimetry of waists" on the unit Euclidean sphere S^{n-1}:

> If $f : S^{k-1} \to S^{n-1}$ is an odd and continuous map, then the $(n-1)$-volume of any ε-neighborhood of $f(S^{k-1})$ in the geodesic distance on the sphere is minimized when f is the canonical embedding, i.e. when the "waist" $f(S^{k-1})$ is an equatorial sphere.

This is proved in [G] for certain k and n and it remains an open problem for the rest of k, n; see next section.

The isoperimetry of waists can be effectively used in the asymptotic convex geometry. Suppose we know that for a symmetric convex body K in $\mathbb{R}^n$ *there exists an orthogonal projection PK that contains the unit Euclidean ball.* Without loss of generality, let S^{k-1} be the sphere of that ball. One can find an odd and continuous lifting $g : S^{k-1} \to K$ of the projection P and contract it to the sphere by defining $f(x) = g(x)/|g(x)|$. Then $f : S^{k-1} \to S^{n-1}$ satisfies the assumptions of Gromov's isoperimetry and, moreover, the waist $f(S^{k-1})$ lies in K. Then the isoperimetry of waists gives a *computable lower bound on the $(n-1)$-volume of any ε-neighborhood of K on the sphere S^{n-1}*. This bound is sharp; it reduces to an equality if the projection PK coincides with the section $K \cap P\mathbb{R}^n$. The exact statement is Proposition 3.1.

This argument is the main step in the proof of (the dual form of) Theorem 1.1. The assumptions are that both K and L have orthogonal projections that contain unit Euclidean balls. The reasoning above based on the isoperimetry of waists implies that the appropriate neighborhoods K_{ε_1} of K and L_{ε_2} of L have large

$(n-1)$-volumes on the unit sphere. Then a standard ε-net argument shows that the Minkowski sum $K_{\varepsilon_1} + UL_{\varepsilon_2}$ contains the unit Euclidean ball with large probability (see Lemma 4.1). If $\varepsilon_1 + \varepsilon_2$ is a small number, then $K + UL$ must contain some nontrivial Euclidean ball, too. This is (the dual form of) the conclusion of Theorem 1.1.

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2 Gromov's isoperimetry of waists

The normalized Lebesgue measure on the unit Euclidean sphere S^n will be denoted by σ_n. For a subset $A \subset S^n$ and a number $\theta > 0$, by A_θ we denote the θ-neighborhood of A in the geodesic distance d, i.e. $A_\theta = \{y \in S^n : \ d(x,y) \leq \theta \text{ for some } x \in A\}$. A map f is called odd is $f(-x) = -f(x)$ for all x. The following isoperimetry is proved in [G] 6.3.B.

Theorem 2.1 (Gromov's isoperimetry of waists). *Let n be odd and $l = 2^k - 1$ for some integer k. Let $f : S^{n-l} \to S^n$ be an odd continuous function. Then for all $0 < \theta < \pi/2$*

$$\sigma_n((f(S^{n-l}))_\theta) \geq \sigma_n((S^{n-l})_\theta).$$

Conjecturally, this theorem should hold for all n, l. We will actually need this for all n, l, and in the absence of such result we will deduce a relaxed version of Gromov's theorem for all n, l, Corollary 2.3 below. This is done naturally by embedding into a higher dimensional sphere.

Lemma 2.2. *Let $A \subset S^n$ be a symmetric measurable set and $m \geq n$ be a positive integer. Then for all $0 < \theta < \pi/2$*

$$\sigma_n(A_\theta) \geq \sigma_m(A_\theta),$$

where in the right side we look at a set A as a subset of S^m via the canonical embedding $S^n \subset S^m$.

Proof. Fix an $x \in S^m$ and let x_1 be its spherical projection onto S^n, i.e. $x_1 = P_n x / |P_n x|$, where P_n denotes the orthogonal projection in $\mathbb{R}^{m+1}$ onto $\mathbb{R}^{n+1}$.

CLAIM: $d(x_1, A) \leq d(x, A)$.

To prove the claim, since A is symmetric it is enough to check that

$$d(x_1, a) \leq d(x, a) \text{ for all } a \in A \text{ such that } d(x, a) \leq \pi/2.$$

Since $0 \le d(x,a) \le \pi/2$ and $\langle x, a\rangle = \cos d(x,a)$, we have

$$0 \le \langle x, a\rangle \le 1.$$

Since $a \in S^n$, we have $P_n a = a$; thus

$$\langle x_1, a\rangle = \langle P_n x/|P_n x|, a\rangle = \frac{1}{|P_n x|}\langle x, a\rangle \ge \langle x, a\rangle.$$

In particular, $0 \le \langle x_1, a\rangle \le 1$. Since the function $\cos^{-1} : [0,1] \to [0, \pi/2]$ is decreasing,

$$d(x_1, a) = \cos^{-1}\langle x_1, a\rangle \le \langle x, a\rangle = d(x,a).$$

This proves the Claim.

Now we can finish the proof of the lemma as follows:

$$\begin{aligned}
\sigma_m(A_\theta) &= \sigma_m(x \in S^m \ : \ d(x, A) \le \theta) \\
&\le \sigma_m(x \in S^m \ : \ d(x_1, A) \le \theta) \\
&= \sigma_n(x_1 \in S^n \ : \ d(x_1, A) \le \theta) = \sigma_n(A_\theta)
\end{aligned}$$

where the last line is obtained by representing a uniformly distributed vector $x \in S^m$ as $x = \gamma x_1 + \sqrt{1-\gamma}x_2$, where $x_1 \in S^n$ and $x_2 \in S^{m-n}$ are uniformly distributed, γ is an appropriate random variable and the three random variables x_1, x_2, γ are jointly independent. ■

Corollary 2.3 (General (relaxed) isoperimetry of waists). *Let $l < n$ are positive integers. Let $f : S^{n-l} \to S^n$ be a n odd continuous function. Then for all $0 < \theta < \pi/2$*

$$\sigma_n((f(S^{n-l}))_\theta) \ge \sigma_{n+l+1}((S^{n-l-1})_\theta).$$

Proof.

CASE 1: $n - l$ is even.

Let k be the minimal integer such that $2^k - 1 \ge l$. Then $m := (n-l) + (2^k - 1)$ is odd. Moreover, since $2^{k-1} < l$, we have $2^k \le 2(l+1)$, so $m < n - l + 2(l+1) - 1 \le n + l + 1$. Hence

$$n \le m \le n + l.$$

Then Gromov's theorem can be applied to functions from $S^{n-l} \to S^m$, in particular to $f : S^{n-l} \to S^n \to S^m$ where the second map is the canonical embedding. Then using Lemma 2.2, Gromov's theorem and Lemma 2.2 again, we have

$$\sigma_n(f(S^{n-l})_\theta) \ge \sigma_m(f(S^{n-l})_\theta) \ge \sigma_m(S^{n-l}_\theta) \ge \sigma_{n+l}(S^{n-l}_\theta).$$

CASE 2: $n-l$ is odd.

Apply Case 1 to the function $g : S^{n-l-1} \to S^{n-l} \to S^n$ where the first map is the canonical embedding and the second map is f. We have

$$\sigma_n(f(S^{n-l})_\theta) \geq \sigma_n(g(S^{n-l-1})_\theta) \geq \sigma_{n+l+1}(S_\theta^{n-l-1}).$$

Therefore for all $l < n$ we have

$$\sigma_n(f(S^{n-l})_\theta) \geq \sigma_{n+l+1}(S_\theta^{n-l-1})$$

(here we used Lemma 2.2 again to step one dimension up in Case 1). ■

To simplify the use of Corollary 2.3, we will denote:

$$\sigma_{n,k}(\theta) = \sigma_n((S^k)_\theta), \quad \sigma_{n,k}^{\mathrm{Lip}}(\theta) = \inf_f \sigma_n((f(S^k))_\theta),$$

where the infimum is over all symmetric continuous functions $f : S^k \to S^n$.

Corollary 2.3′. *Let $k < n$ be positive integers. Then for $0 < \theta < \pi/2$*

$$\sigma_{n,k}^{\mathrm{Lip}}(\theta) \geq \sigma_{2n-k+1,k-1}(\theta). \tag{2.1}$$

Remark. If Gromov's theorem is true for all n, l, then Corollary 2.3′ improves to

$$\sigma_{n,k}^{\mathrm{Lip}}(\theta) \geq \sigma_{n,k}(\theta).$$

The right hand side of (2.1) is a computable quantity. Sharp asymptotic estimates on $\sigma_{n,k}(\theta)$ were found by S.Artstein [A]. For our present purpose, we will be satisfied with less precise estimates, which reduce to computations on Gaussians and whose prove we include for completeness.

Lemma 2.4. *Let $1 < k \leq n$ be integers and let $0 < \varepsilon < 1/2$. Then*

$$(c\varepsilon)^{2k} \leq \sigma_{n-1,n-k-1}(\sin^{-1}\sqrt{\frac{\varepsilon^2 k}{n}}) \leq (C\varepsilon)^{k/2}.$$

Consequently,

$$1-(C\varepsilon)^{k/2} \leq \sigma_{n-1,k-1}(\sin^{-1}\sqrt{1-\frac{\varepsilon^2 k}{n}}) \leq 1-(c\varepsilon)^k.$$

For the proof, we quote two known facts about the canonical real Gaussian vector.

Fact 2.5. *Let $g_1, g_2, \dots$ be a sequence of i.i.d. normalized Gaussian random variables. Then*

(i) For every $M \geq 2$ one has

$$\mathbb{P}\{g_1^2 + \cdots + g_k^2 > M^2 k\} \leq 2e^{-cM^2 k};$$

(ii) for every $\varepsilon > 0$, we have

$$(c\varepsilon)^k \leq \mathbb{P}\{g_1^2 + \cdots + g_k^2 \leq \varepsilon^2 k\} \leq (C\varepsilon)^k.$$

Proof of Lemma 2.4. (i) By the rotation invariance of the Gaussian density,

$$\sigma := \sigma_{n-1,n-k-1}(\sin^{-1}\sqrt{\frac{\varepsilon^2 k}{n}}) = \mathbb{P}\{(g_1^2 + \cdots + g_k^2) \leq \frac{\varepsilon^2 k}{n}(g_1^2 + \cdots + g_n^2)\}.$$

If we write $\frac{\varepsilon^2 k}{n} = \frac{\varepsilon^4 k}{\varepsilon^2 n}$ then by Fact 2.5 (ii) we have

$$\sigma \geq \mathbb{P}\{g_1^2 + \cdots + g_k^2 \leq \varepsilon^4 k\} - \mathbb{P}\{g_1^2 + \cdots + g_n^2 \leq \varepsilon^2 n\} \geq (c\varepsilon^2)^k - (C\varepsilon)^n \geq (c_1\varepsilon)^{2k}$$

since $1 < k < n/4$.

To prove the reverse inequality, let $M \geq 2$ and write $\frac{\varepsilon^2 k}{n} = \frac{M^2\varepsilon^2 k}{M^2 n}$. Then by Fact 2.5 (i) and (ii) we have

$$\sigma \leq \mathbb{P}\{g_1^2+\cdots+g_k^2 \leq M^2\varepsilon^2 k\} - \mathbb{P}\{g_1^2+\cdots+g_n^2 > M^2 n\} \leq (CM\varepsilon)^k + 2e^{-cM^2 n}. \quad (2.2)$$

If $e^{-4cn} \leq (2C\varepsilon)^k$ then letting $M = 2$ in (2.2) we obtain $\sigma \leq 3(C\varepsilon)^k$, as required. Thus we can assume that $e^{-4cn} > (2C\varepsilon)^k > \varepsilon^{4k}$, hence $\log(1/\varepsilon) \geq (cn/k)$. Let $M = 2\sqrt{(k/cn)\log(1/\varepsilon)}$. Note that $2 < M \leq C_1\sqrt{\log(1/\varepsilon)}$. With this M in (2.2) we obtain

$$\sigma \leq (C_2\varepsilon\sqrt{\log(1/\varepsilon)})^k + 2\varepsilon^{-4k} \leq (C_3\varepsilon)^{k/2}.$$

This proves the first part of the Lemma. The second part follows from the equation

$$\sigma_{n-1,n-k-1}(\sin^{-1}\alpha) + \sigma_{n-1,k-1}(\sin^{-1}\sqrt{1-\alpha^2}) = 1, \quad 0 < \alpha < 1,$$

and the first part. ■

When the general Gromov's theorem (Corollary 2.3′) is combined with Lemma 2.4, we obtain explicit estimates for $\sigma_{n,k}^{\mathrm{Lip}}(\theta)$:

Corollary 2.6. *Let $1 < k \leq n$ be integers and let $0 < \varepsilon < 1/2$. Then*

(i) $\sigma_{n-1,n-k-1}^{\mathrm{Lip}}(\sin^{-1}\sqrt{\frac{\varepsilon^2 k}{n}}) \geq (c\varepsilon)^{8k}$;

(ii) $\sigma_{n-1,k-1}^{\mathrm{Lip}}(\sin^{-1}\sqrt{1-\frac{\varepsilon^2 k}{n}}) \geq 1-(C\varepsilon)^{k/4}$.

Proof. Let $\alpha = k/n$.

(i) By Corollary 2.3′,

$$\sigma^{\text{Lip}}_{n-1,(1-\alpha)n-1}(\sin^{-1}\sqrt{\varepsilon^2\alpha}) \geq \sigma_{n+\alpha n,(1-\alpha)n-2}(\sin^{-1}\sqrt{\varepsilon^2\alpha}). \tag{2.3}$$

To apply Lemma 2.4, write the right hand side of (2.3) for suitable m and β as

$$\sigma_{m-1,(1-\beta)m-1}(\sin^{-1}\sqrt{(\varepsilon^2\alpha/\beta)\cdot\beta}) \geq (c\sqrt{\varepsilon^2\alpha/\beta})^{2\beta m}. \tag{2.4}$$

The numbers m and β are, of course, determined by $m-1 = n+\alpha n$ and $(1-\beta)m-1 = (1-\alpha)n-2$. Hence $\beta = (2\alpha n+2)/(n+\alpha n+1)$, so that $\alpha < \beta < 3\alpha$. Then we can continue (2.4) as

$$\geq (c_1\varepsilon)^{4(\alpha n+1)} \geq (c_1\varepsilon)^{8k}.$$

This completes part (i).

(ii) By Corollary 2.3′,

$$\sigma^{\text{Lip}}_{n-1,\alpha n-1}(\sin^{-1}\sqrt{1-\varepsilon^2\alpha}) \geq \sigma_{2n-\alpha n,\alpha n-2}(\sin^{-1}\sqrt{1-\varepsilon^2\alpha}). \tag{2.5}$$

To apply Lemma 2.4, write the right hand side of (2.5) for suitable m and β as

$$\sigma_{m-1,\beta m-1}(\sin^{-1}\sqrt{1-(\varepsilon^2\alpha/\beta)\cdot\beta}) \geq 1-(C\sqrt{\varepsilon^2\alpha/\beta})^{\beta m} - e^{-10m}. \tag{2.6}$$

The numbers m and β are, of course, determined by $m-1 = 2n-\alpha n$ and $\beta m-1 = \alpha n-2$. Hence $\beta = (\alpha n-1)/(2n-\alpha n+1)$, so that $\beta \geq \alpha/2$. Then we can continue (2.6) as

$$\geq 1-(C_1\varepsilon)^{(\alpha n-1)/2} \geq 1-(C_1\varepsilon)^{k/4}.$$

This completes part (ii). ■

3 Waists generated by projections of convex bodies

The following observation connects the isoperimetry of waists to convex geometry.

For simplicity, given a set $A \in \mathbb{R}^n$ we write $\sigma_{n-1}(A)$ for $\sigma_{n-1}(A \cap S^{n-1})$, if measurable. The unit Euclidean ball in $\mathbb{R}^n$ is denoted by D. Minkowski sum in $\mathbb{R}^n$ is defined as $A + B = \{a+b:\ a \in A,\ b \in B\}$.

Proposition 3.1. *Let K be a convex symmetric set in $\mathbb{R}^n$. Assume there is an orthogonal projection P,* $\text{rank}P = k$*, such that $PK \supseteq PD$. Then for all $0 < \varepsilon < 1$*

$$\sigma_{n-1}(K+\varepsilon D) \geq \sigma^{\text{Lip}}_{n-1,k-1}(\sin^{-1}\varepsilon). \tag{3.1}$$

Remark. The power of this fact is that the right side of (3.1) is easily estimated via Gromov's theorem (Corollary 2.6).

Proof. We can assume that the range of P is $\mathbb{R}^k$, so $PK \supseteq S^{k-1}$. There exists an odd continuous lifting $g : S^{k-1} \to K$ of the projection P. Define

$$f : S^{k-1} \to S^{n-1}, \quad f(x) = g(x)/|g(x)|.$$

The function f is odd and it is continuous because

$$|g(x)| \geq |Pg(x)| = |x| = 1 \quad \text{for all } x.$$

Since also $g(x) \in K$, we have $f(x) \in K$, thus

$$f(S^{k-1}) \subseteq K \cap S^{n-1}.$$

By making a simple planar drawing, one sees that for every $y \in S^{n-1}$

$$[-y, y] + \varepsilon D \supseteq \{y\}_{\sin^{-1}\varepsilon} \tag{3.2}$$

Running y over $f(S^{k-1})$, we obtain

$$\begin{aligned} K + \varepsilon D &\supseteq f(S^{k-1}) + \varepsilon D \\ &= \bigcup_{y \in f(S^{k-1})} \Big([-y, y] + \varepsilon D \Big) \quad \text{by the symmetry of } f \\ &= f(S^{k-1})_{\sin^{-1}\varepsilon} \quad \text{by (3.2).} \end{aligned}$$

Intersecting both sides with S^{n-1} and taking the measure completes the proof. ■

Proposition 3.1 will in particular be used to estimate the covering number of $K + \varepsilon D$.

Given two convex sets L and K, the covering number $N(L, K)$ is the minimal number of translates of K needed to cover L. By a simple and known volumetric argument, $N(L, K) \leq \frac{|L+K|}{|K|}$.

Lemma 3.2. *For every convex symmetric set K,*

$$N(D, K) \leq 2^n / \sigma_{n-1}(K).$$

Proof.

$$N(D, K) \leq N(D, K \cap D) \leq \frac{|D + (K \cap D)|}{|K \cap D|} \leq \frac{|2D|}{|K \cap D|}. \tag{3.3}$$

Next,

$$\sigma_{n-1}(K) = \sigma_{n-1}(K \cap D) \le \frac{|K \cap D|}{|D|}, \tag{3.4}$$

which folows from a standard argument that transfers the surface measure on S^{n-1} to the volume in D (a set $A \subseteq S^{n-1}$ generates the cone $\cup_{0<t<1} tA$, which occupies the same portion of the volume in D as $\sigma_{n-1}(A)$).

Then (3.3) and (3.4) complete the proof. ■

4 Proof of Theorem 1.1

By duality, Theorem 1.1 can equivalently be stated as follows. There exist an absolute constant $a \in (0,1)$ such that the following holds. Assume that there exist orthogonal projections P and Q with $\text{rank}P = k$ and $\text{rank}Q = n - ak$, and such that

$$PK \supseteq PD, \quad QL \supseteq QD. \tag{4.1}$$

Then for U as in the theorem, we claim that

$$K + UL \supseteq C^{n/k} D. \tag{4.2}$$

The idea is as follows. Let $\delta_K, \delta_L > 0$ be parameters. By Gromov's theorem and Lemma 3.2, we will be able to estimate

$$1 - \sigma := \sigma_{n-1}(K + \delta_K D) \quad \text{and} \quad N := N(2D, L + \delta_L D). \tag{4.3}$$

Lemma 4.1. *Let K and L be convex bodies in $\mathbb{R}^n$ such that (4.3) holds and $\delta_K + \delta_L < 1$. Then for a random orthogonal operator $U \in \mathcal{O}(n)$*

$$(1 - \delta_K - \delta_L) D \subseteq K + UL \tag{4.4}$$

with probability at least $1 - N\sigma$.

Proof. By a standard argument, the sphere S^{n-1} of D can be covered by N translates of the body $L + \delta_L D$ by vectors from S^{n-1}. Hence there exists a subset $\mathcal{N} \subset S^{n-1}$ such that

$$|\mathcal{N}| = N \quad \text{and} \quad D \subseteq \mathcal{N} + L + \delta_L D. \tag{4.5}$$

Since for every $z \in S^{n-1}$, its image Uz under a random rotation $U \in \mathcal{O}(n)$ is uniformly distributed on the sphere, we have for any fixed $z \in \mathcal{N}$:

$$\mathbb{P}\{U \in \mathcal{O}(n) \ : \ Uz \in K + \delta_K D\} = \sigma_{n-1}(K + \delta_K D) = 1 - \sigma.$$

Thus

$$\mathbb{P}\{U \in \mathcal{O}(n) \ : \ U\mathcal{N} \subseteq K + \delta_K D\} \geq 1 - N\sigma.$$

Fix any U in this set and apply it to the inclusion in (4.5):

$$D \subseteq U\mathcal{N} + UL + \delta_L D \subseteq K + \delta_K D + UL + \delta_L D.$$

Since $\delta_K + \delta_L < 1$, this inclusion implies (4.4). ■

Proof of Theorem 1.1. We can clearly assume that $0 < a < 1/33$ and that $ak \geq 1$. Let

$$\varepsilon_K > 0, \quad \delta_K = \sqrt{1 - \frac{\varepsilon_K^2 k}{n}}.$$

By Proposition 3.1 and Corollary 2.6 (ii),

$$\sigma_{n-1}(K + \delta_K D) \geq \sigma_{n-1,k-1}^{\mathrm{Lip}}(\sin^{-1}\delta_K) \geq 1 - (C\varepsilon_K)^{k/4} \geq 1 - 2e^{-10n}$$

if one chooses the value of ε_K as

$$\varepsilon_K = \exp(-C_1 n/k),$$

where $C_1 > 0$ is a sufficiently large absolute constant. Similarly, let

$$\varepsilon_L > 0, \quad \delta_L = \sqrt{\frac{\varepsilon_L^2 ak}{n}}.$$

By Proposition 3.1 and Corollary 2.6 (i),

$$\sigma_{n-1}(L + \delta_L D) \geq \sigma_{n-1,n-ak-1}^{\mathrm{Lip}}(\sin^{-1}\delta_L) \geq (c\varepsilon_L)^{8k} \geq \frac{1}{2}e^{-n/2}$$

if one chooses the value of ε_L as

$$\varepsilon_L = \exp(-c_2 n/ak)$$

where $c_2 > 0$ is a sufficiently small absolute constant. By Lemma 3.2,

$$N(2D, 2L + 2\delta_L D) = N(D, L + \delta_L D) \leq 2^n / \frac{1}{2}e^{-n/2} \leq 2e^{1.2n}.$$

By Lemma 4.1, if $\delta_K + 2\delta_L < 1$ then the desired inclusion

$$(1 - \delta_K - 2\delta_L)D \subseteq K + 2UL \tag{4.6}$$

holds with probability at least

$$1 - 2e^{1.2n} \cdot 2e^{-10n} \geq 1 - e^{-n}.$$

So it only remains to bound below

$$\delta_K + 2\delta_L = \sqrt{1 - \exp(-2C_1 n/k)(k/n)} + 2\sqrt{\exp(-2c_2 n/ak)(ak/n)}.$$

This can be quickly done using the inequalities $\sqrt{1-x} \leq 1 - x/2$ and $xe^{-C/x} \geq e^{-2C/x}$ valid for all $0 < x < 1$ and for a sufficiently large absolute constant C. We thus have

$$\delta_K + 2\delta_L \leq 1 - \frac{1}{2}\exp(-C_1' n/k) + 2\exp(-c_2' n/ak) < 1 - \frac{1}{4}\exp(-Cn/k)$$

if a is chosen a sufficiently small absolute constant. This together with (4.6) completes the proof. ∎

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